SHANNON'S SAMPLING THEOREM IN A DISTRIBUTIONAL SETTING

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Abstract. The classical Shannon sampling theorem states that a signal f with Fourier transform $F \in L^2(\mathbb{R})$ having its support contained in $(-\pi, \pi)$ can be recovered from the sequence of samples $(f(n))_{n\in\mathbb{Z}}$ via

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin(\pi(t-n))}{\pi(t-n)} \quad (t \in \mathbb{R}).$$

In this article we prove a generalization of this result under the assumption that $F \in \mathcal{E}'(\mathbb{R})$ has support contained in $(-\pi, \pi)$.

1. Introduction

A signal $f : \mathbb{R} \to \mathbb{R}$ *band limited* if it has a compactly supported Fourier transform. The classical Shannon's sampling theorem says that a band limited signal can be recovered from its samples provided that the sampling frequency is large enough. More precisely, the following result is known:

Theorem 1.1. *Let $F \in L^2(\mathbb{R})$ be such that its support contained in $(-\pi, \pi)$, and let its inverse Fourier transform be f. Then*

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin(\pi(t-n))}{\pi(t-n)} \quad (t \in \mathbb{R}). \tag{1}$$

Although in the signal processing literature, this is referred to as "Shannon's sampling theorem", it was independently discovered earlier among others by Whittaker [4].

The question of the validity of (1) for general compactly supported distributions $F \in \mathcal{E}'(\mathbb{R})$, with support in $(-\pi, \pi)$, is a natural one. One might then hope that the series (1) converges in the sense of distributions. We will show in Example 3.2 that this is not true, rather the following analogue of (1) does hold.

Theorem 1.2 (Sampling theorem)**.** *Suppose that $F \in \mathcal{E}'(\mathbb{R})$ has its support contained in $(-\pi, \pi)$, and let f be the inverse Fourier transform of F. Then we have the following:*

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(1) *If $G \in \mathcal{E}'(\mathbb{R})$ has its support also contained in $(-\pi, \pi)$ and the inverse Fourier transform of g satisfies $g(n) = f(n)$ $(n \in \mathbb{Z})$, then $g = f$ and $G = F$.*

(2) *Furthermore, f can be reconstructed from its samples by the formula*

$$f = \mathcal{F}^{-1}\left(\mathbb{1}_{[-\pi,\pi]} \sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}\right). \tag{2}$$

where the series converges in $\mathcal{S}'(\mathbb{R})$.

Here $\mathbb{1}_{[-\pi,\pi]}$ denotes the indicator function of the interval $[-\pi, \pi]$. Notice that formally, if the multiplication by $\mathbb{1}_{[-\pi,\pi]}$ were to distribute over the infinite sum, and if the resulting series were to converge in $\mathcal{S}'(\mathbb{R})$, then we would indeed obtain (1):

$$\begin{aligned} f &= \mathcal{F}^{-1}\left(\mathbb{1}_{[-\pi,\pi]} \sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}\right) \\ &= \sum_{n\in\mathbb{Z}} f(n)\mathcal{F}^{-1}(\mathbb{1}_{[-\pi,\pi]}e^{-in\omega}) \\ &= \sum_{n\in\mathbb{Z}} f(n)\frac{\sin(\pi(\cdot - n))}{\pi(\cdot - n)}. \end{aligned}$$

So in this sense our result above is an appropriate generalization of the classical sampling theorem.

1.1. **Known results.** In earlier works, for example in [3] and [1], distributional analogues of Theorem 1.1 were obtained. However, in [3], an additional assumption was made on F, namely that in the decomposition of F as

$$F = G^{(k)} + \sum_{\ell=0}^{k-1} c_\ell \delta^{(\ell)}$$

where G is a piecewise continuous function, one has that $G/(\omega^2 - \pi^2)^{k+2} \in L^1(\mathbb{R})$. Also there the question of *pointwise* convergence of (1) was considered, and the series was shown to converge pointwise in the sense of (C, α)-summability; see [3] for the details. In contrast, we do not make this extra assumption on $F \in \mathcal{E}'(\mathbb{R})$ with support in $(-\pi, \pi)$, and ask for *distributional* convergence.

In [1], the extra assumption was absent, and just as in our main result in Theorem 1.2, there too the only assumption on F was that $F \in \mathcal{E}'(\mathbb{R})$ with support in $(-\pi, \pi)$. However, there once again *pointwise* convergence was considered, and the formula (1) was modified in that an extra convergence factor was present. There it was shown that

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\frac{\sin(\pi(t-n))}{\pi(t-n)} S(q\pi(t-n)) \quad (t \in \mathbb{R})$$

where $0 < q < 1$ is such that the support of F is contained in

$$\{\omega \in \mathbb{R} : |\omega| < (1-q)\pi\},$$

and the function S is defined by

$$S(y) = \frac{\displaystyle\int_{-1}^{1} e^{1/(x^2-1)-ixy}\,dx}{\displaystyle\int_{-1}^{1} e^{1/(x^2-1)}\,dx} \quad (y \in \mathbb{R}).$$

So our result is quite different from these earlier works.

2. Proof of the main result

For preliminaries on Distribution Theory, we refer the reader to [2]. We use the standard notation $\mathcal{D}(\mathbb{R})$ for the space of compactly supported test functions, and $\mathcal{D}'(\mathbb{R})$ for its dual space of distributions. The space of compactly supported distributions will be denoted by $\mathcal{E}'(\mathbb{R})$, which can be identified with the dual of the space of smooth functions $\mathcal{E}(\mathbb{R}) = C^\infty(\mathbb{R})$. The space of rapidly decreasing test functions is denoted by $\mathcal{S}(\mathbb{R})$, and its dual space, the Schwartz space of tempered distributions by $\mathcal{S}'(\mathbb{R})$. The translation operation τ_a on distributions is defined by

$$\langle \tau_a T, \varphi \rangle = \langle T, \varphi(\cdot + a) \rangle \quad (\varphi \in \mathcal{D}(\mathbb{R})).$$

$\tau_a : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ is continuous. A distribution $T \in \mathcal{D}'(\mathbb{R})$ is said to be *periodic with a period* $a > 0$ if $T = \tau_a T$. The subspace of $\mathcal{D}'(\mathbb{R})$ consisting of periodic distributions with period a is denoted by $\mathcal{D}'_a(\mathbb{R})$. δ_a denotes the Dirac distribution supported at $a \in \mathbb{R}$, and we set $\delta = \delta_0$.

Proof of Theorem 1.2. Clearly, (1) follows from (2), since

$$f = \mathcal{F}^{-1}\left(\mathbb{1}_{[-\pi,\pi]} \sum_{n\in\mathbb{Z}} f(n) e^{-in\omega}\right) = \mathcal{F}^{-1}\left(\mathbb{1}_{[-\pi,\pi]} \sum_{n\in\mathbb{Z}} g(n) e^{-in\omega}\right) = g.$$

Now we prove the statement (2). As $F \in \mathcal{E}'(\mathbb{R})$, we have

$$f(t) = \frac{1}{2\pi} \langle F, e^{it\cdot} \rangle \quad (t \in \mathbb{R}).$$

We know that f is the restriction to $\mathbb{R}$ of an entire function.

Define $\widetilde{F} \in \mathcal{D}'(\mathbb{R})$ by

$$\widetilde{F} = F * \sum_{n\in\mathbb{Z}} \delta_{2\pi n}.$$

The convolution is well-defined since F has compact support. Then $\widetilde{F}$ belongs to $\mathcal{D}'_{2\pi}(\mathbb{R})$, since the translation operation $\tau_{2\pi}$ is just convolution by $\delta_{2\pi}$.

We expand $\widetilde{F} \in \mathcal{D}'_{2\pi}(\mathbb{R})$ in its Fourier series:

$$\widetilde{F} = \sum_{n\in\mathbb{Z}} \widetilde{F}_n e^{in\omega},$$

where $\widetilde{F}_n = \dfrac{1}{2\pi}\langle F, e^{-in\omega\cdot}\rangle = f(-n)$, and so

$$\widetilde{F} = \sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}, \tag{3}$$

where the series converges in $\mathcal{D}'(\mathbb{R})$.

Also, by the Payley-Wiener-Schwartz theorem, since $F \in \mathcal{E}'(\mathbb{R})$ has its support in $(-\pi, \pi)$, it follows that there exist constants $C, N \geq 0$ such that

$$|f(z)| \leq C(1+|z|)^{-N} e^{\pi|\mathrm{Im}(z)|}, \quad z \in \mathbb{C}.$$

In particular, $|f(n)| \leq C(1+|n|)^{-N}$, and so the series

$$\sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}$$

converges in $\mathcal{S}'(\mathbb{R})$.

We have that the singular support of $\widetilde{F}$ is contained in its support, which is furthermore included in the set

$$\bigcup_{n\in\mathbb{Z}} (n\pi, (n+1)\pi).$$

Also the singular support of $\mathbb{1}_{[-\pi,\pi]}$ is the set $\{-\pi, \pi\}$. Since

$$\text{sing supp } \mathbb{1}_{[-\pi,\pi]} \bigcap \text{sing supp } \widetilde{F} = \emptyset,$$

it follows that $\mathbb{1}_{[-\pi,\pi]}\widetilde{F}$ is well-defined; see [2, Remark 2, p.55]. Thus we obtain

$$F = \mathbb{1}_{[-\pi,\pi]}\widetilde{F} = \mathbb{1}_{[-\pi,\pi]} \sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}.$$

Taking inverse Fourier transforms, we obtain

$$f = \mathcal{F}^{-1}\left(\mathbb{1}_{[-\pi,\pi]} \sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}\right).$$

This completes the proof. $\square$

3. Examples

Example 3.1. Let $F = \delta' \in \mathcal{E}'(\mathbb{R})$. Then

$$f(t) = \frac{t}{2\pi i} \quad (t \in \mathbb{R}).$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$ be nonzero and nonnegative test function such that

$$\text{supp}(\varphi) \subset (1/3, 2/3).$$

Let $m > 0$ be such that $\sin(\pi t) > m$ for $t \in (1/3, 2/3)$. For n odd, we have

$$\begin{aligned}\int_{\mathbb{R}} \frac{\sin(\pi(t-n))}{\pi(t-n)}\varphi(t)dt &= \int_{\mathbb{R}} \frac{\sin(\pi(n-t))}{\pi(n-t)}\varphi(t)dt = \int_{\mathbb{R}} \frac{\sin(\pi t)}{\pi(n-t)}\varphi(t)dt \\ &\geq \int_{\mathbb{R}} \frac{m}{\pi(n-t)}\varphi(t)dt \geq \frac{m}{\pi n}\int_{\mathbb{R}} \varphi(t)dt.\end{aligned}$$

Thus

$$\begin{aligned}\left|\left\langle f(n)\frac{\sin(\pi(\cdot-n))}{\pi(\cdot-n)},\varphi\right\rangle\right| &= n\frac{1}{2\pi}\left|\int_{\mathbb{R}}\frac{\sin(\pi(t-n))}{\pi(t-n)}\varphi(t)dt\right| \\ &\geq n\frac{1}{2\pi}\frac{m}{\pi n}\int_{\mathbb{R}}\varphi(t)dt \not\longrightarrow 0\end{aligned}$$

as $n\to\infty$. So $\displaystyle\sum_{n\in\mathbb{Z}} f(n)\frac{\sin(\pi(\cdot-n))}{\pi(\cdot-n)}$ does not converge in $\mathcal{D}'(\mathbb{R})$. $\Diamond$

Example 3.2. Consider the same example $F=\delta'\in\mathcal{E}'(\mathbb{R})$, and check (2) of Theorem 1.2 in this case. We have

$$\sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}=\sum_{n\in\mathbb{Z}}\frac{n}{2\pi i}e^{-in\omega}=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{d}{d\omega}e^{-in\omega}=\frac{1}{2\pi}\frac{d}{d\omega}\sum_{n\in\mathbb{Z}}e^{-in\omega}.$$

Set

$$T=\sum_{n\in\mathbb{Z}}e^{-in\omega}\in\mathcal{S}'(\mathbb{R}).$$

Then T is 2π-periodic. Also we have $(e^{i\omega}-1)T=0$, and so $\operatorname{supp}(T)\subset 2\pi\mathbb{Z}$. In a neighbourhood N of 0, we can write $e^{i\omega}-1=\omega\alpha(\omega)$ with $\alpha\in\mathcal{E}(\mathbb{R})$ and $\alpha(\omega)\neq 0$ for ω belonging to this neighbourhood N, so that on N, $\omega T=0$. This means that $T=c\delta_0$ in N. As T is 2π-periodic, we obtain

$$T=\sum_{n\in\mathbb{Z}}e^{-in\omega}=c\sum_{n\in\mathbb{Z}}\delta_{2\pi n}.$$

In other words,

$$\mathcal{F}\left(\sum_{n\in\mathbb{Z}}\delta_n\right)=c\sum_{n\in\mathbb{Z}}\delta_{2\pi n}$$

and so for all $\varphi\in\mathcal{S}(\mathbb{R})$,

$$\sum_{n\in\mathbb{Z}}\langle\delta_n,\mathcal{F}\varphi\rangle=c\sum_{n\in\mathbb{Z}}\langle\delta_{2\pi n},\varphi\rangle.$$

But with $\varphi:=e^{\omega^2/4\pi}$, $\mathcal{F}\varphi=2\pi e^{-\pi t^2}$. Substituting this, we obtain that $c=2\pi$. Hence $T=2\pi\sum_{n\in\mathbb{Z}}\delta_{2\pi n}$. So we obtain

$$\sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}=\frac{1}{2\pi}\frac{d}{d\omega}\sum_{n\in\mathbb{Z}}e^{-in\omega}=\frac{1}{2\pi}\frac{d}{d\omega}2\pi\sum_{n\in\mathbb{Z}}\delta_{2\pi n}=\sum_{n\in\mathbb{Z}}\delta'_{2\pi n}.$$

So

$$1_{[-\pi,\pi]}\sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}=1_{[-\pi,\pi]}\sum_{n\in\mathbb{Z}}\delta'_{2\pi n}=\delta'=F.$$

Consequently, $f=\mathcal{F}^{-1}F=\mathcal{F}^{-1}\left(1_{[-\pi,\pi]}\displaystyle\sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}\right)$, as expected. $\Diamond$

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